

Compactification near and on the light front

A. Harindranath^{a,b}, E. Martinovič^{a,c}, and J. P. Vary^{a,d}

^aInternational Institute of Theoretical and Applied Physics,
Iowa State University, Ames, IA 50011, U.S.A.

^bSaha Institute of Nuclear Physics, 1/AF Bidhan Nagar, Calcutta, 700064, India

^cInstitute of Physics, Slovak Academy of Sciences
Dúbravská cesta 9, 842 28 Bratislava, Slovakia

^dDepartment of Physics and Astronomy, Iowa State University, Ames, IA 50011, U.S.A.
(December 9, 1999)

Abstract

We address problems associated with compactification near and on the light front. In perturbative scalar field theory we illustrate and clarify the relationships among three approaches: (1) quantization on a space-like surface close to a light front; (2) infinite momentum frame calculations; and (3) quantization on the light front. Our examples emphasize the difference between zero modes in space-like quantization and those in light front quantization. In particular, in perturbative calculations of scalar field theory using discretized light cone quantization there are well-known new “zero mode induced” interaction terms. However, we show that they decouple in the continuum limit and covariant answers are reproduced. Thus compactification of a light-like surface is feasible and defines a consistent field theory.

I. INTRODUCTION

Problems pertaining to compactification near and on the light front have been presented and explored recently in the context of perturbative scalar field theory [1]. In the formalism of compactification near the light front certain divergences were found in the one loop scattering amplitude in scalar field theory *at finite box length* as one tried to approach the light front. These divergences were presumed to be caused by the longitudinal zero momentum modes in the light front theory.

Zero modes on the light front have a long history [2–4]. For certain field theories, they are invoked to account for the non-trivial vacuum structure. In order to isolate the zero mode, one puts the system in a longitudinal box and imposes boundary conditions. This procedure is popularly known as Discretized Light Cone Quantization [5] (DLCQ). In DLCQ scalar field theory with periodic boundary conditions, the zero modes are constrained and they have to be determined in terms of the non-zero modes by solving a nonlinear operator equation. Thus the zero mode in scalar light front theory is quite different from the zero mode in equal time theory where it is a dynamical mode just as any non-zero mode. It is important to keep this distinction in mind.

Since zero modes pose a major challenge in the nonperturbative context, attempts have been made to perform the quantization on a space like surface [6] close to the light front (a parameter η characterizes the “closeness”). By taking $\eta \rightarrow 0$ one is supposed to reach the light front surface. However, this limiting procedure need not be smooth since a light front surface cannot be reached from a space-like surface by a *finite* Lorentz transformation. On the other hand, S-matrix elements should be independent of η for *any* value of η since this parameterization simply labels different space-like surfaces. Thus any η dependence in an S-matrix element signals breakdown of Lorentz invariance as in the results of Ref. [1].

Let us recall the major differences between the discretized versions of near light front theory and light front theory. We shall restrict the longitudinal coordinate x^- ($x^\pm = x^0 \pm x^3$) to a finite interval while keeping two transverse coordinates unbounded. To avoid confusion, we shall denote the light front box length by L and the near light front box length by L_{et} . In order to check Lorentz invariance one has to perform the continuum limit of DLCQ. Let us consider the mass operator $M^2 = P^+P^- - (P^\perp)^2$, where P^+, P^- are the light-front momentum and energy operators and $P^\perp \equiv (P^1, P^2)$. In DLCQ one can introduce $P^+ = \frac{2\pi}{L}K$ and $P^- = \frac{L}{2\pi}H$. The semi positive definite operator K , the harmonic resolution, is dimensionless momentum and H , the Hamiltonian, has the dimension of M^2 . In DLCQ the mass operator is given by $M^2 = KH - (P^\perp)^2$. The box length L has disappeared from the operator. Eigenvalues of K represent the total momentum of the system. The continuum limit is given by $K \rightarrow \infty$. This is to be contrasted with the near light front discretization where the box length does not disappear from the mass operator. Also the momentum operator is not semi positive definite. Nevertheless for the ease of comparisons, let us denote the total dimensionless momentum in the near light front case by K . The longitudinal momentum P in this case can take both positive and negative values and we can put only $|P| = \frac{2\pi}{L_{et}}K$.

The infinite momentum frame [7,8] is a concept that allows one to simulate perturbative light front theory calculations in an equal time framework by taking the external total longitudinal momentum to infinity. In scalar field theory the equivalence holds even beyond tree level (except for vacuum diagrams). For theories involving fermions the equivalence clearly breaks down beyond tree level [9,10]. In scalar field theory, in the discretized version, one can ask whether one can simulate DLCQ perturbation theory by considering the infinite momentum frame starting from the equal time formulation. Obviously this cannot be achieved by taking K very large since that should correspond to the continuum limit of DLCQ. One choice is to take $L_{et} \rightarrow 0$ since this can simulate infinite momentum for non-zero modes. Then one can ask the question whether L_{et} drops out of scattering amplitudes and if they in turn approach DLCQ scattering amplitudes. Of course by taking $L_{et} \rightarrow 0$ we have moved as far away from the continuum limit as possible and if we find Lorentz non-invariant answers we should not be surprised. Another choice is to discretize the near light front theory, let $\eta \rightarrow 0$ and see whether L_{et} dependence drops out (characteristic of the DLCQ formalism). At finite η , $L \rightarrow \infty$ readily reproduces covariant answers, but at finite L , $\eta \rightarrow 0$ produces divergent answers. From this one cannot conclude anything about DLCQ since Lorentz invariance is broken. Note that in the discretized near light front formulation, where modes are specified by integers n , the expression $\frac{n}{\eta}$, encountered in [1], presents for the zero mode ($n = 0$) a $\frac{0}{0}$ problem for $\eta \rightarrow 0$ which means that the limit is undefined.

To the best of our knowledge, various issues that are raised above have not been resolved

in a clear and satisfactory manner. Towards this goal, we perform and compare perturbative calculations for scalar field theory in the continuum and discretized versions of three formulations, namely, light front quantization, infinite momentum limit of equal time quantization and space-like quantization parameterized by η . As examples we consider the self-energy diagram in ϕ^3 theory and the scattering diagram in ϕ^4 theory.

The plan of this paper is as follows. In Sec. II we discuss the results in the continuum light front theory. In Sec. III the corresponding DLCQ results are presented. In Sec. IV the results in space like quantization characterized by η are given in the continuum and the discretized versions. Corresponding results in the infinite momentum limit of equal time theory are given in Sec. V. Sec. VI contains our summary and conclusions. Since it is unfamiliar to most readers, a brief introduction to the space-like quantization parameterized by η is given in Appendix A.

II. LIGHT FRONT PERTURBATION THEORY – CONTINUUM FORMULATION

In this section we compare results of the light front perturbation theory with those of the covariant perturbation theory, both in continuum formulation. We consider the self-energy diagram in $\frac{\lambda}{3!}\phi^3$ theory and the scattering diagram in $\frac{\lambda}{4!}\phi^4$ theory.

A. Self-energy in $\frac{\lambda}{3!}\phi^3$ theory

Consider the one loop self-energy diagram in ϕ^3 theory. Note that in this case there is only one time ordered diagram (Fig. 1a) in the light front case. Using the rules of light front old fashioned perturbation theory [11] we have

$$\Sigma(p^2) = \frac{1}{2}\lambda^2 \int_0^{p^+} \frac{dq^+ d^2 q^\perp}{2(2\pi)^3} \frac{1}{q^+(p^+ - q^+)} \frac{1}{p^- - \frac{(q^\perp)^2 + m^2}{q^+} - \frac{(p^\perp - q^\perp)^2 + m^2}{p^+ - q^+} + i\epsilon}. \quad (2.1)$$

The factor 1/2 is a symmetry factor. Introducing $y = q^+/p^+$, we get

$$\Sigma(p^2) = \frac{1}{2} \frac{\lambda^2}{2(2\pi)^3} \int_0^1 dy d^2 q^\perp \frac{1}{y(1-y)p^2 - (q^\perp)^2 - m^2 + i\epsilon} \quad (2.2)$$

with $p^2 = p^+p^- - (p^\perp)^2$.

Note that the integrand is nonsingular at $y = 0$.

Next we derive this result starting from the Feynman diagram. The corresponding amplitude (Fig. 2) is

$$-i\Sigma(p^2) = \frac{1}{2} \frac{(-i\lambda)^2}{(2\pi)^4} \int d^4 k \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p+k)^2 - m^2 + i\epsilon}. \quad (2.3)$$

Using $d^4 k = \frac{1}{2} dk^+ dk^- d^2 k^\perp$, we have

$$\Sigma(p^2) = -\frac{i}{2} \frac{(-i\lambda)^2}{(2\pi)^4} \frac{1}{2} \int_{-\infty}^{+\infty} dk^+ \int d^2 k^\perp \int_{-\infty}^{+\infty} dk^- \frac{1}{k^+(p^+ + k^+)} \frac{1}{k^- - \frac{(k^\perp)^2 + m^2}{k^+} + i\frac{\epsilon}{k^+}} \frac{1}{p^- + k^- - \frac{(p^\perp + k^\perp)^2 + m^2}{p^+ + k^+} + i\frac{\epsilon}{p^+ + k^+}}. \quad (2.4)$$

Let us now perform the k^- integration. Let $p^+ > 0$. For $k^+ > 0$, $p^+ + k^+ > 0$, both poles are in the lower half of the complex k^- plane. We can close the contour in the upper half plane and the integral is zero. For $k^+ < 0$, if $p^+ < -k^+$, $p^+ + k^+ < 0$, both poles are in the upper half plane. We can close the contour in the lower half plane and the integral again is zero. For $p^+ > 0$, we get a non-vanishing contribution when $k^+ < 0$ and $k^+ > -p^+$. Then, closing the contour in the upper half plane, we get

$$\Sigma(p^2) = \frac{1}{2} \frac{(-i\lambda)^2}{2(2\pi)^3} \int_{-p^+}^0 \frac{dk^+ d^2 k^\perp}{k^+(p^+ + k^+)} \frac{1}{p^- + \frac{(k^\perp)^2 + m^2}{k^+} - \frac{(p^\perp - k^\perp)^2 + m^2}{p^+ + k^+} - i\frac{\epsilon}{k^+} + i\frac{\epsilon}{p^+ + k^+}} \quad (2.5)$$

or

$$\Sigma(p^2) = \frac{1}{2} \lambda^2 \int_0^{p^+} \frac{dq^+ d^2 q^\perp}{2(2\pi)^3} \frac{1}{q^+(p^+ - q^+)} \frac{1}{p^- - \frac{(q^\perp)^2 + m^2}{q^+} - \frac{(p^\perp - q^\perp)^2 + m^2}{p^+ - q^+} + i\epsilon}. \quad (2.6)$$

We recover the expression (Eq. (2.1)) from old fashioned perturbation theory with energy denominator and integration over three momentum.

B. One loop scattering in $\frac{\lambda}{4!}\phi^4$ theory

Next consider the scattering amplitude at one loop level in ϕ^4 theory. p_1, p_2 are the initial momenta and p_3, p_4 are the final momenta. Let us denote $s = (p_1 + p_2)^2$ and $t = (p_1 - p_3)^2$. In the light front perturbation theory, we have to consider two cases separately.

1) $p_1^+ > p_3^+$.

The scattering amplitude (Fig. 3a) in this case is

$$\begin{aligned} M_{fi} &= \frac{1}{2} \frac{\lambda^2}{2(2\pi)^3} \int_0^{p_1^+ - p_3^+} dq_1^+ \int d^2 q_1^\perp \frac{1}{q_1^+} \frac{1}{p_1^+ - p_3^+ - q_1^+} \\ &\quad \frac{1}{p_1^- + p_2^- - p_3^- - p_4^- - q_1^- - (p_1 - p_3 - q_1)^-} \\ &= \frac{1}{2} \frac{\lambda^2}{2(2\pi)^3} \int_0^{p_1^+ - p_3^+} dq_1^+ \int d^2 q_1^\perp \frac{1}{q_1^+} \frac{1}{p_1^+ - p_3^+ - q_1^+} \frac{1}{p_1^- - p_3^- - q_1^- - (p_1 - p_3 - q_1)^-} \\ &= \theta(p_1^+ - p_3^+) \frac{1}{2} \frac{\lambda^2}{2(2\pi)^3} \int_0^1 dy \int d^2 q^\perp \frac{1}{y(1-y)t - (q^\perp)^2 - m^2 + i\epsilon}. \end{aligned} \quad (2.7)$$

We have used the notation $(p - q)^- = \frac{(p^\perp - q^\perp)^2 + m^2}{p^+ - q^+}$.

2) $p_1^+ < p_3^+$.

The scattering amplitude (Fig. 3b) in this case is

$$\begin{aligned}
M_{fi} &= \frac{1}{2} \frac{\lambda^2}{2(2\pi)^3} \int_0^{p_3^+ - p_1^+} dq_2^+ \int d^2 q_2^\perp \frac{1}{q_2^+} \frac{1}{p_3^+ - p_1^+ - q_2^+} \\
&\quad \frac{1}{p_1^- + p_2^- - p_1^- - p_4^- - q_2^- - (p_1 - p_3 - q_2)^-} \\
&= \frac{1}{2} \frac{\lambda^2}{2(2\pi)^3} \int_0^{p_3^+ - p_1^+} dq_2^+ \int d^2 q_2^\perp \frac{1}{q_2^+} \frac{1}{p_3^+ - p_1^+ - q_2^+} \frac{1}{p_3^- - p_1^- - q_2^- - (p_3 - p_1 - q_2)^-} \\
&= \theta(p_3^+ - p_1^+) \frac{1}{2} \frac{\lambda^2}{2(2\pi)^3} \int_0^1 dy \int d^2 q^\perp \frac{1}{y(1-y)t - (q^\perp)^2 - m^2 + i\epsilon}. \tag{2.8}
\end{aligned}$$

We have used overall energy conservation $p_1^- + p_2^- = p_3^- + p_4^-$ and hence $p_2^- - p_4^- = p_3^- - p_1^-$. Adding the two contributions we get

$$M_{fi} = \frac{1}{2} \frac{\lambda^2}{2(2\pi)^3} \int_0^1 dy \int d^2 q^\perp \frac{1}{y(1-y)t - (q^\perp)^2 - m^2 + i\epsilon}. \tag{2.9}$$

Note that the integrand is nonsingular at $y = 0$.

Starting from the Feynman amplitude, the scattering amplitude (Fig. 4) is

$$\begin{aligned}
-iM_{fi} &= \frac{1}{2} \frac{(-i\lambda)^2}{(2\pi)^4} \int d^4 q \frac{i}{q^2 - m^2 + i\epsilon} \frac{i}{(p_1 - p_3 - q)^2 - m^2 + i\epsilon} \\
M_{fi} &= \frac{1}{2} i \frac{\lambda^2}{(2\pi)^4} \frac{1}{2} \int dq^+ d^2 q^\perp dq^- \frac{1}{q^+(p_1^+ - p_3^+ - q^+)} \times \\
&\quad \frac{1}{q^- - \frac{(q^\perp)^2 + m^2}{q^+} + i\frac{\epsilon}{q^+}} \frac{1}{p_1^- - p_3^- - q^- - \frac{(p_1^+ - p_3^+ - q^+)^2}{p_1^+ - p_3^+ - q^+} + i\frac{\epsilon}{p_1^+ - p_3^+ - q^+}}. \tag{2.10}
\end{aligned}$$

Now we have to distinguish two cases separately.

1) $p_1^+ - p_3^+ > 0$.

Non-vanishing contribution can occur only when $q^+ > 0$ and $p_1^+ - p_3^+ - q^+ > 0$. Then poles appear in both upper and lower half planes in the complex k^- plane. Closing the contour in the lower half plane, we get,

$$\begin{aligned}
M_{fi} &= \frac{1}{2} i \frac{\lambda^2}{(2\pi)^4} \frac{1}{2} (-2\pi i) \int_0^{p_1^+ - p_3^+} dq^+ \int d^2 q^\perp \frac{1}{q^+(p_1^+ - p_3^+ - q^+)} \\
&\quad \frac{1}{p_1^- - p_3^- - \frac{(q^\perp)^2 + m^2}{q^+} - \frac{(p_1^+ - p_3^+ - q^+)^2 + m^2}{p_1^+ - p_3^+ - q^+} + i\epsilon}. \tag{2.11}
\end{aligned}$$

2) $p_1^+ - p_3^+ < 0$.

Non vanishing contribution can occur only when $q^+ < 0$ and $p_1^+ - p_3^+ - q^+ < 0$. Closing the contour in the upper half plane, we get

$$\begin{aligned}
M_{fi} &= \frac{1}{2} i \frac{\lambda^2}{(2\pi)^4} \frac{1}{2} (2\pi i) \int_{p_1^+ - p_3^+}^0 dq^+ \int d^2 q^\perp \frac{1}{q^+(p_1^+ - p_3^+ - q^+)} \\
&\quad \frac{1}{p_1^- - p_3^- - \frac{(q^\perp)^2 + m^2}{q^+} - \frac{(p_1^+ - p_3^+ - q^+)^2 + m^2}{p_1^+ - p_3^+ - q^+} + i\epsilon}. \tag{2.12}
\end{aligned}$$

Thus, in this case, we reproduce the two time ordered diagrams in old fashioned perturbation theory (Fig. 3).

After a change of variable in the second contribution, the two contributions can be combined to yield

$$M_{fi} = \frac{1}{2} \frac{\lambda^2}{2(2\pi)^3} \int_0^1 dy \int d^2 q^\perp \frac{1}{y(1-y)t - (q^\perp)^2 - m^2 + i\epsilon}. \quad (2.13)$$

III. LIGHT FRONT PERTURBATION THEORY – DISCRETIZED FORMULATION

Light front quantization in a finite volume with periodic fields (DLCQ) has some conceptual advantages. First of all, it allows one to work explicitly with Fourier modes of quantum fields, carrying vanishing light front momentum p^+ – the zero modes (ZM). While in the case of gauge fields some ZM are dynamically independent, ZM of scalar fields are always dependent (constrained) variables, as follows from the structure of the equations of motion, containing $\partial_\mu \partial^\mu = 4\partial_+ \partial_- - \partial_\perp^2$, $\partial_\perp^2 \equiv \partial_i \partial_i$, $i = 1, 2$. Due to periodic boundary conditions¹ in x^- and $x^\perp \equiv (x^1, x^2)$ ($-L \leq x^- \leq L$, $-L_\perp \leq x^\perp \leq L_\perp$), the full scalar field can be decomposed as $\phi(x) = \phi_0(x^+, x^\perp) + \phi_n(x^+, \underline{x})$, where $\underline{x} \equiv (x^-, x^\perp)$. The mode expansion for the normal-mode field $\phi_n(\underline{x})$ is

$$\phi_n(\underline{x}) = \frac{1}{\sqrt{V}} \sum_{\underline{k}} \frac{1}{\sqrt{k^+}} \left[a_{\underline{k}} e^{-i\underline{k}\underline{x}} + a_{\underline{k}}^\dagger e^{i\underline{k}\underline{x}} \right]. \quad (3.1)$$

Here we have used the notation $\underline{k}\underline{x} \equiv \frac{1}{2}k^+x^- - k^\perp x^\perp$ and $k^+ = \frac{2\pi}{L}n$, $n = 1, 2, \dots, N$, $k^\perp = \frac{2\pi}{L_\perp}n^\perp$, $n^\perp = 0, \pm 1, \pm 2, \dots, \pm N_\perp$. In the following, the integration over the 3-dimensional volume will be denoted by $\int_V d^3 \underline{x} \equiv \frac{1}{2} \int_{-L}^L dx^- \int_{-L_\perp}^{L_\perp} d^2 x^\perp$.

A. ϕ^3 theory

The DLCQ Hamiltonian of the ϕ^3 theory, obtained in the canonical way, is

$$P^- = \int_V d^3 \underline{x} \left[m^2 \phi^2 + (\partial_\perp \phi)^2 + \frac{\lambda}{3} \phi^3 \right]. \quad (3.2)$$

It contains ZM terms, which have to be expressed by means of the normal-mode field $\phi_n(\underline{x})$. To do so we need to obtain the lowest-order solution of the ZM constraint. The latter is simply the ZM projection of the equation of motion

$$(4\partial_+ \partial_- - \partial_\perp^2) \phi = -m^2 \phi - \frac{\lambda}{2} \phi^2 \quad (3.3)$$

¹We use finite interval for all three space coordinates in this section.

and reads

$$(m^2 - \partial_\perp^2)\phi_0 = -\frac{\lambda}{2} \int_{-L}^L \frac{dx^-}{2L} (\phi_0^2 + \phi_n^2). \quad (3.4)$$

It can be solved iteratively and to the lowest order in λ one has

$$\phi_0 = -\frac{\lambda}{2} \frac{1}{m^2 - \partial_\perp^2} \int_{-L}^L \frac{dx^-}{2L} \phi_n^2. \quad (3.5)$$

The symbolic inverse operator $(m^2 - \partial_\perp^2)^{-1}$ is defined in momentum representation by replacing ∂_\perp^2 by the minus square of the perpendicular momentum of the composite operator in the integrand. In the Fock representation, one finds

$$\phi_0(x^\perp) = -\frac{\lambda}{V} \sum_{\underline{k}_1, \underline{k}_2} \frac{\delta_{\underline{k}_1^+, \underline{k}_2^+}}{\sqrt{k_1^+ k_2^+}} \frac{e^{-i(\underline{k}_1^\perp - \underline{k}_2^\perp)x^\perp}}{m^2 + (\underline{k}_1^\perp - \underline{k}_2^\perp)^2} a_{\underline{k}_1}^\dagger a_{\underline{k}_2} - \frac{\lambda}{2m^2} \frac{1}{V} \sum_{\underline{k}_1} \frac{1}{k_1^+}, \quad (3.6)$$

where the second term comes from the normal ordering. This term will be neglected henceforth because it generates divergent terms in the Hamiltonian, which are presumably a manifestation of the well known pathology of the $\lambda\phi^3$ theory (no lower bound of the energy). Indeed, in the case of the $\lambda\phi^4$ interaction, the constrained zero mode is expressed automatically as a normal-ordered product of creation and annihilation operators without a c-number piece.

The interacting Hamiltonian P_{int}^- contains a term, corresponding to the usual one of the continuum formulation, plus the ZM term, calculated to $O(\lambda^2)$:

$$P_{int}^- = P_{NM} + P_{ZM}^{-(2)}, \quad P_{NM} = \frac{\lambda}{3} \int_V d^3\underline{x} \phi_n^3, \quad (3.7)$$

$$P_{ZM}^{-(2)} = \int_V d^3\underline{x} \left[\phi_0(m^2 - \partial_\perp^2)\phi_0 + \frac{\lambda}{3}(\phi_0\phi_n^2 + \phi_n\phi_0\phi_n + \phi_n^2\phi_0) \right] \quad (3.8)$$

with ϕ_0 given by Eq.(3.5). The symmetric operator ordering has been used in the last term. The $O(\lambda^2)$ self-energy amplitude, corresponding to the first term in (3.7), can be calculated by the old fashioned perturbation theory formula

$$T_{fi} = \sum_n \frac{\langle \underline{p}' | P_{NM} | n \rangle \langle n | P_{NM} | \underline{p} \rangle}{p^- - p_n^-}, \quad (3.9)$$

where H_I denotes the interacting Hamiltonian, $|\underline{p}\rangle \equiv a_{\underline{p}}^\dagger |0\rangle$ and the summation runs over the two-particle intermediate states $|n\rangle \equiv 2^{-\frac{1}{2}} a_{\underline{i}_1}^\dagger a_{\underline{i}_2}^\dagger |0\rangle$. After inserting the field expansion (3.1) and performing the operator commutations, we arrive at

$$T_{fi} = \frac{\delta_{\underline{p}, \underline{p}'}}{\sqrt{p^+ V p'^+ V}} \frac{\lambda^2}{4} \sum_{\underline{q}} \frac{1}{q^+ (p^+ - q^+)} \frac{1}{\frac{(p^\perp)^2 + m^2}{p^+} - \frac{(q^\perp)^2 + m^2}{q^+} - \frac{(p^\perp - q^\perp)^2 + m^2}{p^+ - q^+}}, \quad (3.10)$$

where $q^+ < p^+ = 2\pi K L^{-1}$, $|q^\perp| < 2\pi \Lambda_\perp L_\perp^{-1}$ and K, Λ_\perp are integers. From this expression, the continuum answer for the self-energy $\Sigma(p^2)$ (2.1) or (2.2) can be extracted in the infinite

volume limit $K, L, \Lambda_\perp, L_\perp \rightarrow \infty$ (p^+ kept fixed) with $\frac{1}{V}\Sigma_q \rightarrow \frac{1}{(2\pi)^3} \int \frac{dq^+}{2} d^2 q_\perp$, $\frac{V}{2}\delta_{p,\underline{k}} \rightarrow (2\pi)^3 \delta(\underline{p} - \underline{q})$. We recall that Σ corresponds to the invariant amplitude M_{fi} which differs by $(2\pi)^3$ times the kinematical factor (first term in (3.10)) from T_{fi} .

The ZM Hamiltonian does not contribute in the continuum limit. Indeed, the first term in (3.8) to $O(\lambda^2)$ is

$$P_{ZM_1}^{-(2)} = \frac{1}{2} \frac{\lambda^2}{V} \left\{ \sum_{\underline{k}_1 \dots \underline{k}_4} \frac{\delta_{k_1^+, k_2^+} \delta_{k_3^+, k_4^+}}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \delta_{k_1^\perp + k_3^\perp, k_2^\perp + k_4^\perp}^2 \frac{a_{\underline{k}_1}^\dagger a_{\underline{k}_3}^\dagger a_{\underline{k}_2} a_{\underline{k}_4}}{m^2 + (k_1^\perp - k_2^\perp)^2} \right. \\ \left. + \sum_{\underline{k}_1, k_2^\perp} \frac{1}{k_1^{+2}} \frac{a_{\underline{k}_1}^\dagger a_{\underline{k}_1}}{m^2 + (k_1^\perp - k_2^\perp)^2} \right\}. \quad (3.11)$$

The second term of (3.8) has the same structure with the individual coefficients -1 and $-\frac{2}{3}$ instead of the overall $\frac{1}{2}$ and thus the full $O(\lambda^2)$ ZM Hamiltonian is equal to

$$P_{ZM}^{-(2)} = -\frac{1}{2} \frac{\lambda^2}{V} \left\{ \sum_{\underline{k}_1 \dots \underline{k}_4} \frac{\delta_{k_1^+, k_2^+} \delta_{k_3^+, k_4^+}}{\sqrt{k_1^+ k_2^+ k_3^+ k_4^+}} \delta_{k_1^\perp + k_3^\perp, k_2^\perp + k_4^\perp}^2 \frac{a_{\underline{k}_1}^\dagger a_{\underline{k}_3}^\dagger a_{\underline{k}_2} a_{\underline{k}_4}}{m^2 + (k_1^\perp - k_2^\perp)^2} \right. \\ \left. + \frac{1}{3} \sum_{\underline{k}, q^\perp} \frac{1}{k^{+2}} \frac{a_{\underline{k}}^\dagger a_{\underline{k}}}{m^2 + (k^\perp - q^\perp)^2} \right\}. \quad (3.12)$$

Its contribution to the boson self-energy in the first order perturbation theory is

$$\tilde{T}_{fi} = -\frac{1}{6} \frac{\delta_{p', p}}{\sqrt{p^+ V p'^+ V}} \frac{\lambda^2}{p^+} \sum_{q^\perp} \frac{1}{m^2 + (p^\perp - q^\perp)^2}. \quad (3.13)$$

The corresponding M -amplitude vanishes in the continuum limit due to the extra L^{-1} factor (a similar result in the case of $\frac{\lambda}{4!}\phi^4(1+1)$ has been obtained in Ref. [12]:

$$\tilde{\Sigma}(p^+, p^\perp) = -\frac{1}{6} \frac{\lambda^2}{(2\pi)^2} \frac{1}{L} \frac{1}{p^+} \int d^2 q^\perp \frac{1}{m^2 + (p^\perp - q^\perp)^2}. \quad (3.14)$$

In this way, DLCQ calculation yields the correct covariant result for the one-loop self-energy in $\lambda\phi^3$ theory in the infinite-volume limit.

B. ϕ^4 theory

In order to calculate the one-loop scattering amplitude in DLCQ perturbation theory for $(4!)^{-1}\phi^4$ (3+1) model, we again need to derive the light front Hamiltonian with $O(\lambda^2)$ ZM effective interactions. Following the same steps as in the previous subsection with $(3!)^{-1}\lambda\phi^3$ interaction replaced by $(4!)^{-1}\lambda\phi^4$, we find

$$P_{int}^- = \frac{2\lambda}{4!} \int_V d^3 \underline{x} \phi_n^4(\underline{x}) + P_{ZM}^{-(2)}, \quad (3.15)$$

where the second-order ZM Hamiltonian is

$$P_{ZM}^{-(2)} = \int_V d^3 \underline{x} \left[\phi_0 (m^2 - \partial_\perp^2) \phi_0 + \frac{2\lambda}{4!} 4\phi_0 \phi_n^3 \right]. \quad (3.16)$$

In the last term, the symmetric operator ordering between the lowest-order solution of the ZM constraint

$$\phi_0 = -\frac{\lambda}{3!} \frac{1}{m^2 - \partial_\perp^2} \int_{-L}^L \frac{dx^-}{2L} \phi_n^3 \quad (3.17)$$

and ϕ_n^3 is assumed. In the Fock representation, one obtains

$$\begin{aligned} \phi_0(x^\perp) = & -\frac{\lambda}{2} \frac{1}{V^{\frac{3}{2}}} \sum_{\underline{k}_1, \underline{k}_2, \underline{k}_3} \frac{1}{\sqrt{k_1^+ k_2^+ k_3^+}} \frac{\delta_{\underline{k}_1^+, \underline{k}_2^+ + \underline{k}_3^+}}{m^2 + (k_1^\perp - k_2^\perp - k_3^\perp)^2} \\ & \left[a_{\underline{k}_3}^\dagger a_{\underline{k}_2}^\dagger a_{\underline{k}_1} e^{-i(k_3^\perp + k_2^\perp - k_1^\perp)x^\perp} + h.c. \right]. \end{aligned} \quad (3.18)$$

Using the formula (3.9) with $|\underline{p}\rangle \rightarrow |\underline{p}_1, \underline{p}_2\rangle$, $|\underline{p}'\rangle \rightarrow |\underline{p}_3, \underline{p}_4\rangle$ and with four-particle intermediate states, one finds after a lot of algebra for the second-order NM scattering amplitude the expression

$$T_{fi} = \frac{\delta_{\underline{p}_4 + \underline{p}_3, \underline{p}_2 + \underline{p}_1}}{\sqrt{p_4^+ V p_3^+ V p_2^+ V p_1^+ V}} \frac{\lambda^2}{4} \sum_{\underline{q}} \frac{1}{q^+ (p_3^+ - p_1^+ - q^+)} \frac{1}{p_3^- - p_1^- - (p_3 - p_1 - q)^-} + (1 \leftrightarrow 3) \quad (3.19)$$

The continuum-limit invariant scattering amplitude M_{fi} , extracted from (3.19), coincides with the covariant answer (2.9). It follows that for consistency the ZM contribution has to vanish in the continuum limit. That this is indeed the case can be checked in the first-order perturbation theory. In the Fock representation, part of the ZM Hamiltonian relevant for $2 \rightarrow 2$ scattering, takes the form

$$P_{ZM_4}^{-(2)} = -\frac{\lambda^2}{4} \frac{1}{V^2} \sum_{q^\perp} \sum_{\underline{k}_4, \underline{k}_3, \underline{k}_2, \underline{k}_1} \frac{\delta_{\underline{k}_4 + \underline{k}_3, \underline{k}_2 + \underline{k}_1}}{\sqrt{k_4^+ k_3^+ k_2^+ k_1^+}} \frac{1}{k_1^+ - k_2^+} \frac{a_{\underline{k}_3}^\dagger a_{\underline{k}_1}^\dagger a_{\underline{k}_4} a_{\underline{k}_2}}{m^2 + (q^\perp + k_2^\perp - k_1^\perp)^2}. \quad (3.20)$$

The corresponding scattering amplitude is

$$\tilde{T}_{fi} = -\frac{\delta_{\underline{p}_4 + \underline{p}_3, \underline{p}_2 + \underline{p}_1}}{\sqrt{p_4^+ V p_3^+ V p_2^+ V p_1^+ V}} \frac{\lambda^2}{8} \frac{1}{p_3^+ - p_1^+} \sum_{q^\perp} \frac{1}{m^2 + (q^\perp + p_1^\perp - p_3^\perp)^2} + (1 \leftrightarrow 3) \quad (3.21)$$

and the invariant amplitude \tilde{M} indeed vanishes for $L \rightarrow \infty$:

$$\tilde{M}_{fi}(p_3^+ - p_1^+, p_3^\perp - p_1^\perp) = -\frac{\lambda^2}{8(2\pi)^3} \frac{1}{p_3^+ - p_1^+} \frac{1}{L} \int d^2 q^\perp \frac{1}{m^2 + (q^\perp + p_1^\perp - p_3^\perp)^2} + (1 \leftrightarrow 3). \quad (3.22)$$

IV. NEAR LIGHT FRONT OLD FASHIONED PERTURBATION THEORY

A. Continuum version

1. ϕ^3 theory

For the ϕ^3 self-energy, we have, using the formula (A24) from the Appendix A

$$\Sigma(p^2) = \frac{1}{2} \lambda^2 \int_{-\infty}^{+\infty} \frac{dq_- d^2 q^\perp}{(2\pi)^3} \left(\frac{1}{\frac{1}{\eta^2}(E_{on}(p) - E_{on}(q) - E_{on}(p - q)) + i\epsilon} - \frac{1}{\frac{1}{\eta^2}(E_{on}(p) + E_{on}(q) + E_{on}(p - q)) - i\epsilon} \right) \quad (4.1)$$

$$= \Sigma_I(p^2) + \Sigma_{II}(p^2). \quad (4.2)$$

The two contributions correspond to two different time orderings (Figs. 1a and 1b) in old fashioned perturbation theory.

Let us now take the $\eta \rightarrow 0$ limit of these expressions. First consider $\Sigma_I(p^2)$. We have, $\lim_{\eta \rightarrow 0} E_{on}(q) = |q| + \frac{\eta^2(m^2 + (q^\perp)^2)}{2|q|} + \dots$. Without loss of generality, we shall set $p_- > 0$. Then we get

$$\Sigma_I(p^2) = \frac{1}{2} \frac{\lambda^2}{(2\pi)^3} \int_{-\infty}^{+\infty} dq_- d^2 q^\perp \frac{1}{2|q_-|} \frac{1}{2|p_- - q_-|} \frac{1}{\frac{p_-}{\eta^2} - \frac{|q_-|}{\eta^2} - \frac{|p_- - q_-|}{\eta^2} + \frac{m^2 + (p^\perp)^2}{2p_-} - \frac{m^2 + (q^\perp)^2}{2|q_-|} - \frac{m^2 + (p^\perp - q^\perp)^2}{2|p_- - q_-|}}. \quad (4.3)$$

Now we have to distinguish various regions. For $q_- > 0$, $p_- - q_- > 0$, we get

$$\Sigma_I(p^2) = \frac{1}{2} \frac{\lambda^2}{(2\pi)^3} \int_0^{p_-} dq_- \int_{-\infty}^{+\infty} d^2 q^\perp \frac{1}{q_-} \frac{1}{p_- - q_-} \frac{1}{\frac{m^2 + (p^\perp)^2}{2p_-} - \frac{m^2 + (q^\perp)^2}{2(q_-)} - \frac{m^2 + (p^\perp - q^\perp)^2}{2(p_- - q_-)}} + \mathcal{O}(\eta^2) \quad (4.4)$$

which agrees with the light front answer. For $q_- > 0$, $p_- - q_- < 0$, the amplitude scales as η^2 which vanishes as $\eta \rightarrow 0$. For $q_- < 0$, $p_- - q_- > 0$ the amplitude again scales as η^2 and thus vanishes also.

Next we consider $\Sigma_{II}(p^2)$. In the limit $\eta \rightarrow 0$, we get

$$\Sigma_{II}(p^2) = -\frac{1}{2} \lambda^2 \int_{-\infty}^{+\infty} \frac{dq_- d^2 q^\perp}{(2\pi)^3} \frac{1}{2|q_-|} \frac{1}{2|p_- - q_-|} \frac{1}{\frac{1}{\eta^2} \left(p_- + |q_-| + |p_- - q_-| \right) + \frac{m^2 + (p^\perp)^2}{2p_-} + \frac{m^2 + (q^\perp)^2}{2|q_-|} + \frac{m^2 + (p^\perp - q^\perp)^2}{2|p_- - q_-|}}. \quad (4.5)$$

For the three cases namely, (a) $q_- > 0$, $p_- - q_- > 0$, (b) $q_- > 0$, $p_- - q_- < 0$, and (c) $q_- < 0$, $p_- - q_- > 0$, we find that $\Sigma_{II}(p^2)$ scales as η^2 which vanishes in the limit.

Thus we observe that for ϕ^3 self-energy, for finite η there are two time ordered diagrams. As $\eta \rightarrow 0$ the "backward moving" diagram vanishes as η^2 and we get the light front perturbation theory answer. It is important to note that for any value of η , the sum of the two contributions should be independent of η as dictated by Lorentz invariance. However, it is sufficient for our purposes to show η independence in the limit $\eta \rightarrow 0$.

2. ϕ^4 theory

For the scattering in ϕ^4 theory, we have two time ordered diagrams (Figs. 3a and 3b). First consider Fig. 3a. We denote $p_- = p_{1-} - p_{3-}$. We have

$$M_{fi(I)} = \frac{1}{2} \lambda^2 \int_{-\infty}^{+\infty} \frac{dq_- d^2 q^\perp}{(2\pi)^3} \frac{1}{2E_{on}(q)} \frac{1}{2E_{on}(p-q)} \frac{1}{\frac{1}{\eta^2}(E_{on}(p_{1-}) - E_{on}(p_{3-}) - E_{on}(q) - E_{on}(p-q)) + i\epsilon}. \quad (4.6)$$

Now consider the limit $\eta \rightarrow 0$. The energy denominator becomes

$$\frac{1}{\frac{1}{\eta^2}(p_- - |q_-| - |p_- - q_-|) + \frac{m^2 + (p_1^\perp)^2}{2p_1^+} - \frac{m^2 + (p_3^\perp)^2}{2p_3^+} - \frac{m^2 + (q^\perp)^2}{2|q_-|} - \frac{m^2 + (p^\perp - q^\perp)^2}{2|p_- - q_-|}}. \quad (4.7)$$

The analysis proceeds as in the case of $\Sigma_I(p^2)$. We get a non-vanishing contribution which matches the light front perturbation theory answer. For $E_{on}(p_{1-}) < E_{on}(p_{3-})$, i.e., $p_{3-} > p_{1-}$, the analysis proceeds as in the case of $\Sigma_{II}(p^2)$ and the contribution vanishes as η^2 .

Next consider $M_{fi(II)}$ (Fig. 3b). For $E_{on}(p_{1-}) > E_{on}(p_{3-})$, i.e., $p_{3-} < p_{1-}$, the analysis proceeds as in the case of $\Sigma_{II}(p^2)$ and the contribution vanishes. For the case $E_{on}(p_{1-}) < E_{on}(p_{3-})$, i.e., $p_{3-} > p_{1-}$, the analysis proceeds as in the case of $\Sigma_I(p^2)$ and we get a non-vanishing contribution that agrees with the light front answer.

B. Discretized version

1. ϕ^3 theory

Let us consider the first term of ϕ^3 self energy diagram (Fig. 1a). Restricting the longitudinal coordinate to a finite interval, we obtain

$$\Sigma_I(p^2) = \frac{1}{2} \lambda^2 \frac{1}{2L} \sum_n \int \frac{d^2 q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2 + m^2)}} \frac{1}{2\sqrt{(\frac{(j-n)\pi}{L})^2 + \eta^2((p^\perp - q^\perp)^2 + m^2)}} \frac{1}{\frac{1}{\eta^2}(E_i - E_I) + i\epsilon} \quad (4.8)$$

where the energy of the initial (i) and intermediate (I) state is given by

$$E_i = \sqrt{(\frac{j\pi}{L})^2 + \eta^2((p^\perp)^2 + m^2)},$$

$$E_I = \sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2 + m^2)} + \sqrt{(\frac{(j-n)\pi}{L})^2 + \eta^2((p^\perp - q^\perp)^2 + m^2)} \quad (4.9)$$

and the discretized longitudinal momenta are

$$q_- = \frac{n\pi}{L}, \quad p_- = \frac{j\pi}{L}, \quad n, j = 0, \pm 1, \pm 2, \dots \quad (4.10)$$

For $j, n \neq 0$, as $\eta \rightarrow 0$, we get the result independent of η and L .

For $n > j$, the amplitude vanishes as $\eta^2 L^2$ for fixed L . For $n = j = 0$, the amplitude diverges as $\frac{1}{\eta L}$.

For Fig. 1b, we have

$$\begin{aligned} \Sigma_{II}(p^2) = & -\frac{1}{2}\lambda^2 \frac{1}{2L} \sum_n \int \frac{d^2 q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2 + m^2)}} \\ & \frac{1}{2\sqrt{(\frac{(j-n)\pi}{L})^2 + \eta^2((p^\perp - q^\perp)^2 + m^2)}} \\ & \frac{1}{\frac{1}{\eta^2}(E_i + E_I) - i\epsilon}. \end{aligned} \quad (4.11)$$

For $j, n \neq 0$, as $\eta \rightarrow 0$, the amplitude vanishes as $\eta^2 L^2$. For $n = j = 0$, the amplitude diverges as $\frac{1}{L\eta}$. It is not difficult to understand the origin of this divergence. We have already seen that there is no dynamical scalar zero mode on the light front and thus the sum over intermediate states cannot include this mode. On the other hand, for arbitrarily small but non-zero η (space-like quantization) there is a dynamical zero mode in the sum over intermediate states. By requiring this state to exist in the limit we are not approaching the light front theory but some peculiar (divergent) regime of the space-like theory. The light-front theory has its own mechanisms (constraints for zero modes) to replace this “missing” dynamical mode.

2. ϕ^4 theory

The scattering amplitude in the discretized form for Fig. 3a reads

$$\begin{aligned} M_{fi(I)} = & \frac{1}{2}\lambda^2 \frac{1}{2L} \sum_n \int \frac{d^2 q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2 + m^2)}} \\ & \frac{1}{2\sqrt{(\frac{(j-k-n)\pi}{L})^2 + \eta^2((p_1^\perp - p_3^\perp - q^\perp)^2 + m^2)}} \\ & \frac{1}{\frac{1}{\eta^2}(E_i - E_I) + i\epsilon} \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} E_i = & \sqrt{(\frac{j\pi}{L})^2 + \eta^2((p_1^\perp)^2 + m^2)} - \sqrt{(\frac{k\pi}{L})^2 + \eta^2((p_3^\perp)^2 + m^2)}, \\ E_I = & \sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2 + m^2)} + \sqrt{(\frac{(j-k-n)\pi}{L})^2 + \eta^2((p_1^\perp - p_3^\perp - q^\perp)^2 + m^2)} \end{aligned} \quad (4.13)$$

and the discretized longitudinal momenta are

$$q_- = \frac{\pi}{L}n, \quad p_{1-} = \frac{\pi}{L}j, \quad p_{3-} = \frac{\pi}{L}k, \quad n, j = 0, \pm 1, \pm 2, \dots \quad (4.14)$$

For Fig. 3b, we get

$$M_{fi(II)} = \frac{1}{2}\lambda^2 \frac{1}{2L} \sum_n \int \frac{d^2 q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2 + m^2)}} \frac{1}{2\sqrt{(\frac{(k-j-n)\pi}{L})^2 + \eta^2((p_3^\perp - p_1^\perp - q^\perp)^2 + m^2)}} \frac{1}{\frac{1}{\eta^2}(E_i - E_I) + i\epsilon}, \quad (4.15)$$

where

$$E_i = \sqrt{(\frac{k\pi}{L})^2 + \eta^2((p_3^\perp)^2 + m^2)} - \sqrt{(\frac{j\pi}{L})^2 + \eta^2((p_1^\perp)^2 + m^2)},$$

$$E_I = \sqrt{(\frac{n\pi}{L})^2 + \eta^2((q^\perp)^2 + m^2)} + \sqrt{(\frac{(k-j-n)\pi}{L})^2 + \eta^2((p_3^\perp - p_1^\perp - q^\perp)^2 + m^2)}. \quad (4.16)$$

As in the case of ϕ^3 theory, for $j - k \neq 0$, $n \neq 0$ there is no problem, but for $j - k = 0$, $n = 0$, we run into divergences.

V. INFINITE MOMENTUM FRAME APPROACH

A. Continuum version

1. ϕ^3 theory

For the ϕ^3 self energy, (Figs. 1a and 1b), using the rules of old fashioned perturbation theory, we obtain

$$\Sigma(p^2) = \frac{1}{2}\lambda^2 \int_{-\infty}^{+\infty} \frac{d^3 q}{(2\pi)^3} \frac{1}{2E_q} \frac{1}{2E_{p-q}} \left(\frac{1}{E_p - E_q - E_{p-q} + i\epsilon} - \frac{1}{E_p + E_q + E_{p-q} - i\epsilon} \right). \quad (5.1)$$

$$= \Sigma_I(p^2) + \Sigma_{II}(p^2). \quad (5.2)$$

Here $E_p = \sqrt{p^2 + (p^\perp)^2 + m^2}$. For ease of notation we have denoted the third component of the three-vector \mathbf{p} as p . The two contributions correspond to two different time orderings in old fashioned perturbation theory. To facilitate the infinite momentum limit, we parametrize the internal momenta as follows: $\mathbf{q} = (xp, q^\perp)$, $\mathbf{p} - \mathbf{q} = ((1-x)p, p^\perp - q^\perp)$. It is important to note that the range of x is $-\infty < x < +\infty$. Now $\frac{d^3 q}{2E_q} = \frac{p dx d^2 q^\perp}{2E_q}$. Let us now take the infinite momentum, $p \rightarrow \infty$ limit of these expressions. It follows that $\frac{p dx}{2E_q} \rightarrow \frac{1}{2} \frac{dx}{|x|}$, $E_q \rightarrow |x| p + \frac{m^2 + (q^\perp)^2}{2|x|p}$.

First consider $\Sigma_I(p^2)$. We get

$$\Sigma_I(p^2) = \frac{1}{2} \frac{\lambda^2}{(2\pi)^3} \int_{-\infty}^{+\infty} dx d^2 q^\perp \frac{1}{2x} \frac{1}{2|1-x|p} \frac{1}{p(1-|x| - |1-x|) + \frac{m^2+(p^\perp)^2}{2p} - \frac{m^2+(q^\perp)^2}{2|x|p} - \frac{m^2+(p^\perp-q^\perp)^2}{2|1-x|p}}. \quad (5.3)$$

Now we have to distinguish various regions. For $x \geq 0$, $1-x \geq 0$, we get

$$\Sigma_I(p^2) = \frac{1}{2} \frac{\lambda^2}{(2\pi)^3} \int_0^1 dx \frac{1}{\frac{m^2+(p^\perp)^2}{2} - \frac{m^2+(q^\perp)^2}{2x} - \frac{m^2+(p^\perp-q^\perp)^2}{2(1-x)}} \quad (5.4)$$

which agrees with the light front answer. For $x > 0$, $1-x < 0$, the amplitude scales as $\frac{1}{p^2}$ which vanishes as $p \rightarrow \infty$. For $x < 0$, $1-x > 0$ the amplitude again scales as $\frac{1}{p^2}$ and vanishes in the limit.

Next we consider $\Sigma_{II}(p^2)$. In the limit $p \rightarrow \infty$, we get

$$\Sigma_{II}(p^2) = -\frac{1}{2} \frac{\lambda^2}{(2\pi)^3} \int_{-\infty}^{+\infty} dx \frac{1}{2|x|} \frac{1}{2|1-x|p} \frac{1}{(p(1+|x| + |1-x|) + \frac{m^2+(p^\perp)^2}{2p} + \frac{m^2+(q^\perp)^2}{2|x|p} + \frac{m^2+(p^\perp-q^\perp)^2}{2|1-x|p})}. \quad (5.5)$$

For the three cases namely, (a) $x > 0$, $1-x > 0$, (b) $x > 0$, $1-x < 0$, and (c) $x < 0$, $1-x > 0$, we find that $\Sigma_{II}(p^2)$ scale as $\frac{1}{p^2}$ which vanishes in the limit.

Thus in old fashioned perturbation theory in the infinite momentum limit ($p \rightarrow \infty$), the “backward going diagram” vanishes as $\frac{1}{p^2}$ in accordance with Weinberg’s results [8].

2. ϕ^4 theory

For the scattering in ϕ^4 theory, we have two time ordered diagrams (Figs. 3a and 3b). Let us start with the first one. We denote $p = p_1 - p_3$. We have

$$M_{fi(I)} = \frac{1}{2} \lambda^2 \int_{-\infty}^{+\infty} dq d^2 q^\perp \frac{1}{(2\pi)^3} \frac{1}{2E_q} \frac{1}{2E_{p-q}} \frac{1}{(E_{p_1} - E_{p_3} - E_q - E_{p-q}) + i\epsilon}. \quad (5.6)$$

Now consider the limit $p \rightarrow \infty$. The energy denominator becomes

$$\frac{1}{p((x_1 - x_3) - |x| - |x_1 - x_3 - x|) + \frac{m^2+(p_1^\perp)^2}{2x_1 p} - \frac{m^2+(p_3^\perp)^2}{2x_3 p} - \frac{m^2+(q^\perp)^2}{2|x|p} - \frac{m^2+(p^\perp-q^\perp)^2}{2|x_1-x_3-x|p}}. \quad (5.7)$$

The analysis proceeds as in the case of $\Sigma_I(p^2)$. We get a non-vanishing contribution which matches the corresponding part of the total light front answer, Eq. (2.7), for $E_{p_1} > E_{p_3}$, i.e.,

$x_1 > x_3$. For $E_{p_1} < E_{p_3}$, i.e., $x_1 < x_3$, the analysis proceeds as in the case of $\Sigma_{II}(p^2)$ and the contribution vanishes as $\frac{1}{p^2}$.

Next consider $M_{fi(II)}$.

$$M_{fi(II)} = -\frac{1}{2}\lambda^2 \int_{-\infty}^{+\infty} \frac{dq d^2 q^\perp}{(2\pi)^3} \frac{1}{(E_{p_1} - E_{p_3} + E_q + E_{p-q}) - i\epsilon}. \quad (5.8)$$

For $E_{p_1} > E_{p_3}$, i.e., $x_1 > x_3$, the analysis proceeds as in the case of $\Sigma_{II}(p^2)$ and the contribution vanishes. For the case $E_{p_1} < E_{p_3}$, i.e., $x_3 > x_1$, the analysis proceeds as in the case of $\Sigma_I(p^2)$ and we get a non-vanishing contribution that agrees with the corresponding contribution of the light front answer, Eq. (2.8). Thus the total result, Eq. (2.9), is obtained.

B. Discretized version

1. ϕ^3 theory

As before, we restrict the longitudinal coordinate to a finite interval. Specifically, we set $-L < x^3 < +L$. The longitudinal momenta $q^3 \rightarrow q_n^3 = \frac{\pi}{L}n$, $n = 0, \pm 1, \pm 2, \dots$. The field operator becomes

$$\phi(x) = \frac{1}{\sqrt{2L}} \sum_n \int d^2 q^\perp \frac{1}{\sqrt{2\omega_n}} \left[a_n(q^\perp) e^{-i\omega_n t + i \frac{n\pi x^3}{L} + i q^\perp \cdot x^\perp} + a_n^\dagger(q^\perp) e^{i\omega_n t - i \frac{n\pi x^3}{L} - i q^\perp \cdot x^\perp} \right]. \quad (5.9)$$

Let us consider the ϕ^3 self energy. For the external momentum we set $p = (\frac{i\pi}{L}, p^\perp)$. We get

$$\begin{aligned} \Sigma(p^2) = & \frac{1}{2}\lambda^2 \frac{1}{2L} \sum_n \int \frac{d^2 q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + (q^\perp)^2 + m^2}} \frac{1}{2\sqrt{(\frac{(j-n)\pi}{L})^2 + (p^\perp - q^\perp)^2 + m^2}} \\ & \left(\frac{1}{\sqrt{(\frac{j\pi}{L})^2 + (p^\perp)^2 + m^2} - \sqrt{(\frac{n\pi}{L})^2 + (q^\perp)^2 + m^2} - \sqrt{(\frac{(j-n)\pi}{L})^2 + (p^\perp - q^\perp)^2 + m^2}} \right. \\ & \left. - \frac{1}{\sqrt{(\frac{j\pi}{L})^2 + (p^\perp)^2 + m^2} + \sqrt{(\frac{n\pi}{L})^2 + (q^\perp)^2 + m^2} + \sqrt{(\frac{(j-n)\pi}{L})^2 + (p^\perp - q^\perp)^2 + m^2}} \right). \end{aligned} \quad (5.10)$$

If we take the continuum limit, then $\frac{1}{2L} \sum_n \rightarrow \frac{dq}{2\pi}$ and we obtain the result of the previous subsection. Then, taking the infinite momentum limit, the second contribution drops out and we get the light front answer from the first contribution alone.

Suppose one takes the limit $L \rightarrow 0$, which is the opposite of the continuum limit $L \rightarrow \infty$. This is an attempt to simulate DLCQ results in a space like box. We do not expect the result to agree with the continuum limit of DLCQ which agrees with covariant perturbation theory results.

For $n \neq 0, j \neq 0, n < j$, in the limit $L \rightarrow 0$, the amplitude becomes independent of L . For $j = n = 0$, the amplitude diverges like $\frac{1}{L}$. For $n > j$ the amplitude vanishes like L^2 . But none of these results have anything to do with either continuum or DLCQ results.

2. ϕ^4 theory

The scattering amplitude for Fig. 3a is given by

$$M_{fi(I)} = \frac{1}{2} \lambda^2 \frac{1}{2L} \sum_n \int \frac{d^2 q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + (q^\perp)^2 + m^2}} \frac{1}{2\sqrt{(\frac{(j-k-n)\pi}{L})^2 + (p_1^\perp - p_3^\perp - q^\perp)^2 + m^2}} \frac{1}{E_i - E_I + i\epsilon}, \quad (5.11)$$

where

$$E_i = \sqrt{(\frac{j\pi}{L})^2 + (p_1^\perp)^2 + m^2} - \sqrt{(\frac{k\pi}{L})^2 + (p_3^\perp)^2 + m^2} \\ E_I = \sqrt{(\frac{n\pi}{L})^2 + (q^\perp)^2 + m^2} + \sqrt{(\frac{(j-k-n)\pi}{L})^2 + (p_1^\perp - p_3^\perp - q^\perp)^2 + m^2}. \quad (5.12)$$

and the discretized longitudinal momenta are

$$q_n^3 = \frac{\pi}{L} n, \quad p_{1j}^3 = \frac{\pi}{L} j, \quad p_{3k}^3 = \frac{\pi}{L} k, \quad j, k, n = \pm 1, \pm 2, \dots \quad (5.13)$$

For Fig. 3b we find

$$M_{fi(II)} = \frac{1}{2} \lambda^2 \frac{1}{2L} \sum_n \int \frac{d^2 q^\perp}{(2\pi)^2} \frac{1}{2\sqrt{(\frac{n\pi}{L})^2 + (q^\perp)^2 + m^2}} \frac{1}{2\sqrt{(\frac{(k-j-n)\pi}{L})^2 + (p_3^\perp - p_1^\perp - q^\perp)^2 + m^2}} \frac{1}{E_i - E_I + i\epsilon}, \quad (5.14)$$

where

$$E_i = \sqrt{(\frac{k\pi}{L})^2 + (p_3^\perp)^2 + m^2} - \sqrt{(\frac{j\pi}{L})^2 + (p_1^\perp)^2 + m^2} \\ E_I = \sqrt{(\frac{n\pi}{L})^2 + (q^\perp)^2 + m^2} + \sqrt{(\frac{(k-j-n)\pi}{L})^2 + (p_3^\perp - p_1^\perp - q^\perp)^2 + m^2}. \quad (5.15)$$

As in the case of ϕ^3 theory, for $j - k \neq 0$, $n \neq 0$, there is no problem but for $j - k = 0$, and $n = 0$, we run into divergences.

VI. SUMMARY AND CONCLUSIONS

In this work, we have studied continuum and discretized versions of the time ordered (“old fashioned”) perturbation theory in the light-front, near light front and infinite momentum frameworks, applied to scalar field theory self-energy and scattering amplitudes. We have recalled important features of the covariant perturbation theory, namely that when Feynman amplitudes are rewritten in terms of the light front variables and the contour integration in the light-front energy complex plane is performed, the Feynman amplitudes reduce back to

the continuum light front answers [14]. Also, as stressed already by Weinberg in 1966 [8], the light front perturbation theory (old-fashioned perturbation theory in a reference frame with "infinite-momentum" at that time) is more economical in the sense that one does not need to introduce Feynman parameters to combine propagators in the corresponding integrals, and the four-dimensional Euclidean integration is replaced by a two-dimensional one. Feynman parameters appear in the light front formulation naturally as light front longitudinal momentum fractions.

More specifically, after demonstrating that the continuum light front perturbation theory has no problem with zero modes and its results agree with the covariant results, we have analyzed the continuum limit of the light-front perturbation theory formulated in a finite volume with periodic fields (DLCQ method). This investigation was motivated by claims [1] that DLCQ is ill-defined since it is divergent when formulated as a limit of the space-like quantization on a hypersurface close to the light front [15]. In this connection, we have first shown that the DLCQ perturbation theory is consistent, because parts of the perturbative amplitudes due to the effective interactions induced by the constrained zero mode vanish in the infinite-volume limit and the covariant results are reproduced. Second, when one considers the light front limit ($\eta \rightarrow 0$) of the near light front discretized amplitudes, the zero-mode contribution indeed diverges for fixed box length. But this disagrees with the light front answer and actually cannot tell anything about the light front zero modes. The point is that the light front zero mode is not dynamical in the scalar theory and thus it is not present in the complete set of intermediate states. By letting $\eta \rightarrow 0$ one is forcing the dynamical space-like zero mode to exist on the light front which leads to an incorrect, diverging amplitude. Recall the light front dispersion relation for a free particle (dynamical quantum) $p^- = (m^2 + (p^\perp)^2)(p^+)^{-1}$ which gives divergent energy for a mode with $p^+ = 0$ (and non-zero numerator). It is remarkable how the light front theory copes with this problem: most of the light front zero modes are non-dynamical, i.e. constrained variables. Thus the above dispersion relation is not applicable. A few known dynamical zero modes are massless and have vanishing p^\perp (global zero modes). The dispersion relation is again not applicable and one has to treat these modes in a different manner (in terms of zero-mode coherent states, e.g. [13].)

In other words, the $\eta \rightarrow 0$ limit does not lead us to the light front theory but to a peculiar non-covariant regime of the discretized space-like theory.

On the other hand, the continuum version of the near light front old fashioned perturbation theory reproduces the light front answers (which agree with the covariant ones). But since this formulation has no particular advantages there is no real reason to use it in practical calculations.

In the continuum version of old fashioned perturbation theory in the infinite momentum frame, there is no problem in perturbation theory. In the box, the zero mode contribution scales as $\frac{1}{L}$ whereas nonzero modes scales independent of L . Thus in perturbation theory zero modes decouple in the continuum limit ($L \rightarrow \infty$).

However, if one reinterprets the boost as the equal time box length $L_{et} \rightarrow 0$ as is done in M-theory literature (to simulate DLCQ for finite L), zero mode contributions diverge. In this limit, however, there are infinite number of terms in the DLCQ Hamiltonian.

We would like to emphasize that our conclusions are based on the careful analysis of the *scalar* field theories. This choice was motivated by the discussion in the M-theory literature

[1], where one-loop scattering in $\frac{\lambda}{4!}\phi^4$ scalar theory was analyzed in the near light front framework. It would be very instructive to perform an analysis, similar to that done in the present work, within a theory with fermions, and in particular in a gauge theory. This would help to clarify some unresolved issues in the light front literature [16]. For example, some authors, using DLCQ perturbation theory, found unpleasant quadratic divergences within the 3+1 dimensional light-front Yukawa model. However, the continuum limit was not studied in their work.

In conclusion, our answer to the question, raised in the paper [1] : Does it makes sense to put a quantum system to a light-like box ? is: *yes, it does*. Light-like compactification is feasible and DLCQ is consistent (there is no problem neither with causality [17]). The discretized (compactified) formulation of the theory on the light-like surface does exist as a straightforward light front field theory, but *not* as a limit of a space-like compactification.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. Department of Energy, Grant No. DE-FG02-87ER40371, Division of High Energy and Nuclear physics, by the NSF Grant No. INT-9515511, by the VEGA Grant No. 2/5085/1998 and by the International Institute of Theoretical and Applied Physics, Iowa State University, Ames, Iowa, U.S.A.

APPENDIX A: FIELD THEORY IN NEAR LIGHT FRONT COORDINATES

1. Kinematics

Consider the set of coordinates

$$\begin{aligned}x^+ &= \frac{1}{\sqrt{2}}(x^0 + x^3) + \frac{1}{2}\eta^2 \frac{1}{\sqrt{2}}(x^0 - x^3) \\x^- &= \frac{1}{\sqrt{2}}(x^0 - x^3) \\x^\perp &= (x^1, x^2).\end{aligned}\tag{A1}$$

We shall take x^+ to be the time variable. Then x^- is a longitudinal coordinate. The metric tensor in the new coordinate system is given by

$$\begin{aligned}\tilde{g}_{\mu\nu} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -\eta^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\ \tilde{g}^{\mu\nu} &= \begin{bmatrix} \eta^2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.\end{aligned}\tag{A2}$$

Thus we have,

$$x^2 = \tilde{g}_{\mu\nu} x^\mu x^\nu = 2x^+ x^- - \eta^2 (x^-)^2 - (x^\perp)^2 = \tilde{g}^{\mu\nu} x_\mu x_\nu = \eta^2 (x_+)^2 + 2x_+ x_- - (x_\perp)^2. \quad (\text{A3})$$

Furthermore,

$$x_+ = x^-, x_- = x^+ - \eta^2 x^-. \quad (\text{A4})$$

The scalar product is $k \cdot x = k^+ x^- + k^- x^+ - \eta^2 k^- x^- - k^\perp \cdot x^\perp = k_+ x^+ + k_- x^- - k_\perp \cdot x_\perp$. Thus k_+ which is conjugate to x^+ is the energy and k_- which is conjugate to x^- is the longitudinal momentum. It is important to keep in mind that $-\infty < k_- < +\infty$.

For an on mass-shell particle of mass m , $k^2 = m^2$ yields

$$\eta^2 (k_+)^2 + 2k_+ k_- - (k^\perp)^2 - m^2 = 0 \quad (\text{A5})$$

which leads to the dispersion relation

$$k_+ = \frac{-k_- \pm \sqrt{(k_-)^2 + (m^2 + (k^\perp)^2)\eta^2}}{\eta^2}. \quad (\text{A6})$$

For an on mass-shell particle, since $k^0 > k^3$, $k^0 > 0$ implies $k_+ > 0$ and hence the Lorentz invariant phase space factor

$$\begin{aligned} \frac{d^4 k}{(2\pi)^4} 2\pi \delta(k^2 - m^2) \theta(k_+) &= \frac{dk_+ dk_- d^2 k^\perp}{(2\pi)^3} \delta(\eta^2 (k_+)^2 + 2k_+ k_- - (k^\perp)^2 - m^2) \theta(k_+) \\ &= \frac{dk_- d^2 k^\perp}{(2\pi)^3 2E_{on}}, \end{aligned} \quad (\text{A7})$$

where $E_{on}(k) = \sqrt{(k_-)^2 + \eta^2((k^\perp)^2 + m^2)}$.

2. Free scalar field theory

Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 = \frac{1}{2} \eta^2 \partial_+ \phi \partial_+ \phi + \partial_+ \phi \partial_- \phi - \frac{1}{2} \partial^\perp \phi \cdot \partial^\perp \phi - \frac{1}{2} m^2 \phi^2. \quad (\text{A8})$$

The equation of motion

$$(\partial_\mu \partial^\mu + m^2) \phi = 0 \quad (\text{A9})$$

becomes

$$(\eta^2 \partial_+ \partial_+ + 2\partial_+ \partial_- - (\partial^\perp)^2 + m^2) \phi = 0. \quad (\text{A10})$$

The general solution is

$$\begin{aligned} \phi(x) &= \int \frac{d^4 k}{(2\pi)^4} f(k) 2\pi \delta(k^2 - m^2) e^{ik \cdot x}, \\ \phi(x) &= \int \frac{dk_- d^2 k^\perp}{(2\pi)^3 2E_{on}(k)} [a(k) e^{i(k_{+on} x^+ + k_- x^- - k^\perp \cdot x^\perp)} + a^*(k) e^{-i(k_{+on} x^+ + k_- x^- - k^\perp \cdot x^\perp)}]. \end{aligned} \quad (\text{A11})$$

In the quantum theory we have

$$\phi(x) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk_- d^2 k^\perp}{2E_{on}(k)} [a(k) e^{i(k_{+on}x^+ + k_-x^- - k^\perp \cdot x^\perp)} + a^\dagger(k) e^{-i(k_{+on}x^+ + k_-x^- - k^\perp \cdot x^\perp)}]. \quad (A12)$$

The conjugate momentum is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \partial_+ \phi} = \eta^2 \partial_+ \phi + \partial_- \phi, \quad (A13)$$

$$\begin{aligned} \pi(x) = & -i \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dk_- d^2 k^\perp}{2E_{on}(k)} E_{on}(k) [a(k) e^{i(k_{+on}x^+ + k_-x^- - k^\perp \cdot x^\perp)} \\ & - a^\dagger(k) e^{-i(k_{+on}x^+ + k_-x^- - k^\perp \cdot x^\perp)}]. \end{aligned} \quad (A14)$$

We have,

$$[\phi(x), \pi(y)]_{x^+=y^+} = i\delta(x^- - y^-)\delta^2(x^\perp - y^\perp), \quad (A15)$$

provided

$$\begin{aligned} [a(k), a^\dagger(k')] &= (2\pi)^3 2E_{on}(k) \delta(k_- - k'_-) \delta^2(k^\perp - k'^\perp), \\ [a(k), a(k')] &= 0, [a^\dagger(k), a^\dagger(k')] = 0. \end{aligned} \quad (A16)$$

The Hamiltonian density is

$$\mathcal{H} = \pi \partial_+ \phi - \mathcal{L} = \frac{1}{2} \frac{(\pi - \partial_- \phi)^2}{\eta^2} + \frac{1}{2} \partial^\perp \phi \cdot \partial^\perp \phi + \frac{1}{2} m^2 \phi^2 \quad (A17)$$

and the Hamiltonian in the Fock representation takes the form

$$H = \int dx^- d^2 x^\perp \mathcal{H} = \int \frac{dk_- d^2 k^\perp}{(2\pi)^3 2E_{on}(k)} \frac{-k_- + \sqrt{(k_-)^2 + (m^2 + (k^\perp)^2)\eta^2}}{\eta^2} a^\dagger(k) a(k). \quad (A18)$$

The propagator is given by

$$\begin{aligned} iS_B(x) &= \langle 0 | T(\phi(x)\phi(0)) | 0 \rangle = \theta(x^+) \langle 0 | \phi(x)\phi(0) | 0 \rangle + \theta(-x^+) \langle 0 | \phi(0)\phi(x) | 0 \rangle, \\ &= \frac{1}{(2\pi)^3} \int \frac{dk_- d^2 k^\perp}{2E_{on}(k)} [\theta(x^+) e^{-i(k_{+on}x^+ + k_-x^- - k^\perp \cdot x^\perp)} + \theta(-x^+) e^{i(k_{+on}x^+ + k_-x^- - k^\perp \cdot x^\perp)}]. \end{aligned} \quad (A19)$$

Using

$$\theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dy e^{iyx} \frac{1}{y - i\epsilon} \quad (A20)$$

and changing integration variables, we get

$$\begin{aligned} iS_B(x) &= \frac{1}{(2\pi)^4} \int dk_+ dk_- d^2 k^\perp e^{i(k_+x^+ + k_-x^- - k^\perp \cdot x^\perp)} \frac{i}{\eta^2(k_+)^2 + 2k_+k_- - (k^\perp)^2 - m^2 + i\epsilon}, \\ iS_B(x) &= \frac{1}{(2\pi)^4} \int d^4 k e^{ik \cdot x} \frac{i}{k^2 - m^2 + i\epsilon}. \end{aligned} \quad (A21)$$

3. Old fashioned perturbation theory

We have the perturbative formula for the S matrix:

$$S_{fi} = \delta_{fi} - 2\pi i \delta(p_{+(on)f} - p_{+(on)i}) T_{fi}, \quad (\text{A22})$$

$$T_{fi} = \langle f | V_S | i \rangle + \sum_n \frac{\langle f | V_S | n \rangle \langle n | V_S | i \rangle}{p_{+(on)i} - p_{+(on)n} + i\epsilon} + \dots \quad (\text{A23})$$

where the on-shell energy $p_{+(on)} = \frac{-p_- + \sqrt{(p_-)^2 + \eta^2(m^2 + (p^\perp)^2)}}{\eta^2}$. Since the longitudinal momentum p_- is conserved at the vertex, we get,

$$T_{fi} = \langle f | V_S | i \rangle + \sum_n \frac{\langle f | V_S | n \rangle \langle n | V_S | i \rangle}{\frac{1}{\eta^2}(E_{(on)i} - E_{(on)n}) + i\epsilon} + \dots \quad (\text{A24})$$

where $E_{on}(p) = \sqrt{(p_-)^2 + \eta^2(m^2 + (p^\perp)^2)}$. The sum over intermediate states $\sum_n \rightarrow \int \frac{dk_- d^2 k^\perp}{(2\pi)^3 2E_{on}}$.

REFERENCES

- [1] S. Hellermann and J. Polchinski, Phys. Rev. D**59** , 125002 (1999).
- [2] T. Maskawa and K. Yamawaki, Prog. Theor. Phys. **56**, 270 (1976).
- [3] V. A. Franke, Yu. V. Novozhilov and E. V. Prokhvatilov, Lett. Math. Phys. **5**, 239, 437 (1981).
- [4] For a review, see E. Martinovič, in *Light Front Quantization and Non-Perturbative QCD*, J.P. Vary and F. Wölz (eds.), published and distributed by the International Institute of Theoretical and Applied Physics, Iowa State University, Ames, IA 50011, U.S.A. (1997). This book is freely accessible on the internet via the WWW at the URL: <http://www.iitap.iastate.edu>.
- [5] T. Maskawa and K. Yamawaki, Ref. [2]
C. B. Thorn, Phys. Rev. D**17**, 1073 (1978);
H.C. Pauli and S.J. Brodsky, Phys. Rev. D**32**, 1993, 2001 (1985).
- [6] E.V. Prokhvatilov and V. A. Franke, Sov. J. Nucl. Phys. **49**, 688 (1989);
F. Lenz, M. Thies, S. Levit, and K. Yazaki, Ann. Phys. (N.Y.) **208**, 1 (1991);
H.W.L. Naus, H.J. Pirner, T.J. Fields, and J.P. Vary, Phys. Rev. D**56**, 8062 (1997).
- [7] S. Fubini and G. Furlan, Physics, **1**, 229 (1964); A detailed discussion of the infinite momentum limit can be found in V. de Alfaro, S. Fubini, G. Furlan, and G. Rosetti, *Currents in Hadron Physics* (North-Holland, Amsterdam, 1973).
- [8] S. Weinberg, Phys. Rev. **150**, 1313, (1966).
- [9] S.D. Drell, D.J. Levy and T.M. Yan, Phys. Rev. D**1** , 1035 (1970).
- [10] A. Harindranath and R.J. Perry, Phys. Rev. D**43**, 492 (1991).
- [11] J.B. Kogut and D.E. Soper, Phys. Rev. D**1**, 2901 (1970).
- [12] T. Heinzl, C. Stern, E. Werner and B. Zeller mann, Zeit. Phys. C **72**, 353 (1996).
- [13] E. Martinovič, hep-th/9811182.
- [14] S.J. Chang and S.K. Ma, Phys. Rev. **180**, 1506 (1969).
- [15] In the context of M-theory, on defining DLCQ of quantum field theory as a light like limit, some recent references are D. Bigatti and L. Susskind, *Review of Matrix Theory*, hep-th/9712072; A. Güijosa, Nucl. Phys. **B533**, 406 (1998); A. Bilal, Nucl. Phys. **B521**, 202 (1998); N. Seiberg and E. Witten, *String Theory and Non-Commutative Geometry*, hep-th/9908142; A. Bilal, *Remarks on defining the DLCQ of Quantum Field Theory as a Light-Like Limit*, hep-th/9909220.
- [16] M. Burkardt, Advances in Nuclear Physics **23**, 1 (1996).
- [17] D. Chakrabarti, A. Mukherjee, R. Kundu and A. Harindranath, hep-th/9910108.

Figures

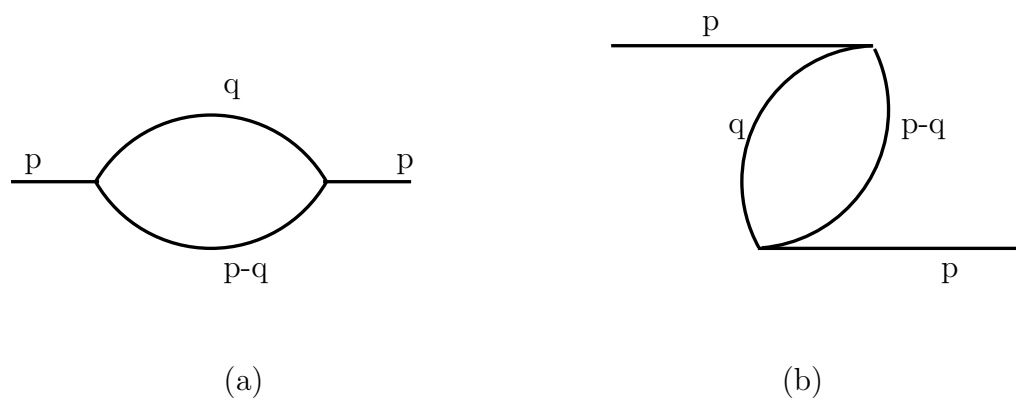


Fig. 1. ϕ^3 self-energy diagrams in old fashioned perturbation theory

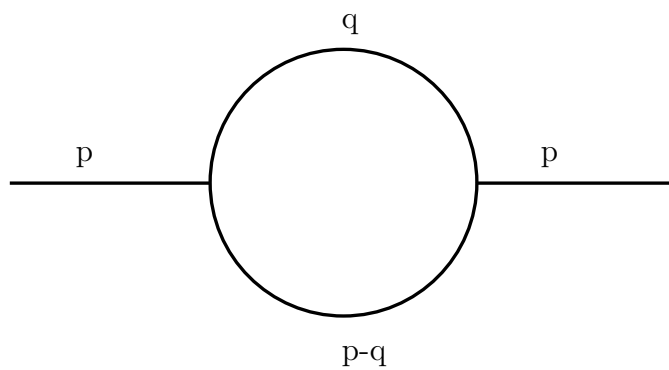
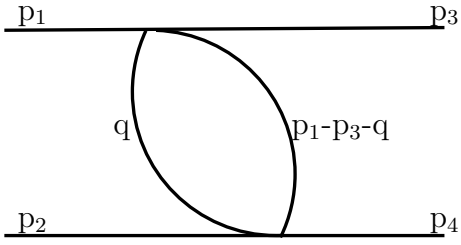
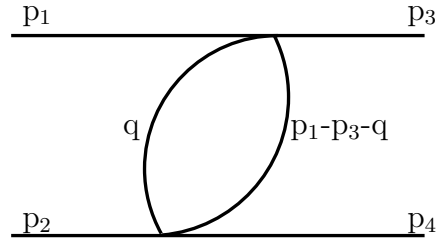


Fig. 2. ϕ^3 self-energy diagram



(a)



(b)

Fig. 3. ϕ^4 scattering diagrams in old fashioned perturbation theory

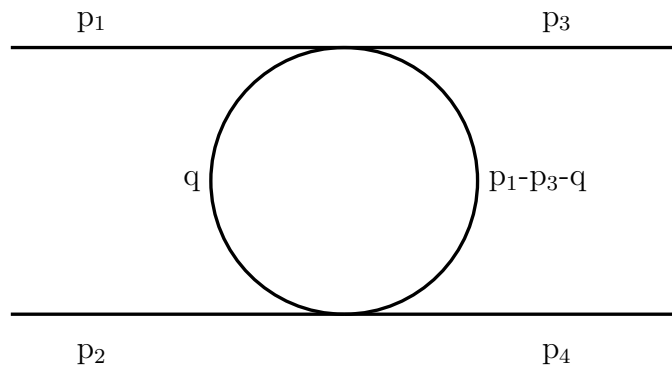


Fig. 4. ϕ^4 scattering diagram